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Mathematical discourse in PhD theses: Considerations for the language teacher

Lee Ming Cherk
Centre for English Language Communication, National University of Singapore

Mathematical discourse is unique in that it is made up of "semiotic resources of mathematical symbolism, visual display in the form of graphs and diagrams, and language" (O'Halloran, 1998, p. 360). Natural language plays the important role of guiding the reader through the logical reasoning of the propositional content. This study investigates how logical reasoning processes are expressed and how authors interact with readers. The study used as data publications on mathematical writing, interviews with disciplinary experts, and an analysis of eight PhD theses in mathematics using Hyland's (2005) metadiscourse framework. The expert writing and interviews showed that a high premium is placed on mathematical content, clarity, conciseness and a sense of collegiality all of which contribute to the aesthetic value of an argument. The analysis of the theses found that mathematical writing at doctoral writing involves both linear mathematical reasoning and meta-mathematical explanations (as defined by Kuteeva & McGrath, 2015). The study also reveals that contrary to the common perception that mathematical writing is impersonal, writers do interact and engage with readers in various ways.

Key Words: metadiscourse; mathematical writing; doctoral academic writing; Singapore

Introduction

Mathematical discourse is made up of "semiotic resources of mathematical symbolism, visual display in the form of graphs and diagrams, and language…. These codes alternate as the primary resource for meaning, and also interact with each other to construct meaning " (O'Halloran, 1998, p. 360). Language plays an important role in guiding the reader through the propositional content and helps the writer to engage the reader, express a stance, and align the reader to the academic community's culture and values.

The fact that epistemological processes influence academic discourse patterns has been well documented in the literature (Becher and Trowler, 2001; Hyland 2005). Mathematics is carried out deductively and mathematical results are shown in proofs established through logical reasoning which ends with a conclusion of either "true" or "false" (Kuteeva & McGrath, 2015; McGrath & Kuteeva, 2012). McGrath and Kuteeva (2012) discussed stance and engagement in pure mathematics research articles, Kuteeva and McGrath (2015) analysed the rhetorical moves in pure math research articles, O'Halloran (1998) investigated classroom discourse in mathematics, and Anthony and Bowen (2013), used a corpus based research approach to study the language of mathematics. In the field of mathematics education, however, there has been increasing interest in the linguistic aspects of mathematical writing (Morgan, Craig, Schuette, & Wagner, 2014). For example, Burton and Morgan (2000) investigated the epistemological perspectives of research mathematicians, and Morgan (1996) studied the role and function

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of language in mathematics education. Such research typically deals with short texts while extended texts like PhD mathematics theses are relatively unexplored.

The research reported here contributes to filling this gap and is expected to inform teachers of postgraduate writing for PhD students of mathematics. The paper will analyse how discourse features are employed in mathematical exposition in PhD theses; and investigate how awareness of discourse features can potentially translate into teaching and learning of mathematical writing. The research attempts to answer three questions:

1. What is "good" mathematical writing, according to the mathematics discourse community?
   This question refers to writing aligned with the tradition, culture and values of the discipline, and deemed fit for the award of a PhD degree. Francis Su (2015), former president of the Mathematical Association of America, considers such writing as: interacting with readers (projecting a textual "voice", engaging the audience with an invitational tone, observing the mathematical culture and etiquette) and clarity.

2. How do metadiscourse elements deliver "good" mathematical writing?
   Hyland's (2005) metadiscourse was used to analyse the two main aspects of mathematical writing: the linear and peripheral logical reasoning in mathematics, and the writer's interactions with the audience specifically through projecting the textual voice, engaging with the audience, and observing the culture and etiquette of mathematics. It should be noted that Martin and White's (2005) APPRAISAL framework was not used because it centres on stance which features significantly less in mathematical writing. Biber's (1988) Multidimensional Analysis was not used because it requires a larger sample than that available.

3. In what ways can awareness of mathematical discourse help PhD students in their writing?
   This question focuses on how metadiscourse awareness can enable students to write in a way that guides their readers through their logical reasoning.

Methodology

This study first surveyed published materials about mathematical writing. Then, email interviews were conducted with mathematics academics and a corpus of eight mathematics PhD theses was analysed.

Review of publications on mathematical writing

The literature review was conducted to understand what mathematicians considered good mathematical discourse and encompassed a range of genres (textbooks, publications by professional bodies and university math department websites) because literature on PhD mathematical writing is lacking (see Appendix 1 for the full list of sources). These texts were written by subject specialists and mathematics educators to inform novice writers how epistemological processes are explained and what constitutes good mathematical discourse.

Interviews with expert writers

Interviews were conducted with four mathematics professors, all published authors who had studied, researched and taught in either Hong Kong, China or Singapore, and were therefore very familiar with the Asian context and the problems faced by Asian writers of mathematics. These professors described good mathematical writing, and then commented on the importance of good writing skills and English instructors' role in promoting it among mathematics students (See Appendix 2 for the questions asked). Follow up questions were asked where necessary.

Corpus study of eight PhD theses

A linguistic perspective was provided through a corpus study which built on an earlier investigation (Lee, 2016). Eight mathematics PhD theses were selected from those submitted to the National University of Singapore Library on the basis that they exhibited typical epistemological processes leading to proofs and contained ancillary meta-mathematical information.

The theses were coded according to the categories laid out in Hyland's (2005) metadiscourse model (see Table 1 for an illustration of the main and sub-metadiscoursal categories, with examples). The researcher was the only coder but the coding was verified by a published author on metadiscourse with a PhD in the area. After coding, the numbers were tallied to provide a profile of the metadiscourse use in mathematical writing.

Table 1. Examples of metadiscourse features in mathematical writing based on Hyland's (2005) metadiscourse model

Category	Function	Examples
Interactive features: Help to guide the reader through the text		
Transitions	Express relations between the main clauses	Hence; if and only if; moreover; then; since; given that X is…, then y…. . ; thus; therefore; however; moreover; to this end; In addition; first, next, then; similarly; on the other hand
Frame markers	Refer to discourse acts; sequences or stages	In this part, we will prove that… ; Now we are ready to combine the respective estimates for each of the five factors; The next section contains some applications and some miscellaneous remarks.
Endophoric markers	Refer to information in other parts of the text	Recall equation X; see figure Y; We looked at some applications of the theory developed in the last section; Recall in chapter 3, we studied…
Evidentials	Refer to information on other texts	A precise lower bound is obtained by P. Yang and M. Zhu [28] using a symmetrisation technique. As in [8], [11] and [25], we consider the following set of…
Code gloss	Elaborate propositional meaning	In other words; In fact; which implies …such that… ; that is; equation (3.6) immediately implies…; We deduce from (4.10) that there exists a sequence…

Interactional features: Involve the reader in the text		
Hedges	Withhold commitment and open dialogue	One may need to develop new apriori estimates
Boosters	Emphasize certainty or close dialogue	This is clearly …; one can easily see that; indeed
Attitude markers	Express writer's attitude to proposition	The situation is trivial; what is important to us is that…; It is interesting to try a flow approach
Self-mentions	Explicit reference to author(s)	One finds; one sees; one obtains;
Engagement markers	Explicitly build relationship with reader	We are going to show that; we need to; let us assume; we have the following; observe that; let; suppose; define; write; check; fix; compute; identity; calculate; show; state; note

Findings

Published writing and mathematics experts

Content

Published works on how to write mathematics (Halmos, 1973; Higham, 1998; Krantz, 1997; Su, 2015), university math department websites (Reiter, 1995; Tomforde, 2007), and transcriptions of a series of lectures on mathematical writing (Knuth, Larrabee, & Roberts, 1987) all dictate that good writing should pay attention to content. This view concurs with that of the four interviewees.

The famed mathematician Halmos states the importance of content in the most definitive terms:

> To have something to say is by far the most important ingredient of good exposition--- so much so that if the idea is important enough, the work has a chance to be immortal even if it is confusingly misorganized and awkwardly expressed. (Halmos, 1973, p. 21)

The academics interviewed for this research hold similar views:

> … the author should write what he/she means and mean what he/she writes. It is essential that the author understands the theory well enough before starting the paper, including even the most trivial details. (Expert 1)

> First of all, the mathematics must be correct. This may not be what you are asking for, but to me the mathematical content is more important. (Expert 2)

> The most important thing in mathematical writing to me is the idea. (Expert 3)

Clarity

Su (2015) states that apart from valuable content, good mathematical writing prioritises communicating exposition with clarity. Halmos (1973), while emphasizing the importance of content, also indicates that to write well, "… you must organize what you want to say, and you must arrange it in the order you said it in" (p. 20). One of expert writers also acknowledged the importance of clear communication:

> The most important word for me is "clarity"…., the structure of the paper should be carefully planned, and each part (…) should play its role, but no more. The author should show mercy on the reader and ensure that statements, assumptions and symbols are stated clearly with no ambiguity. (Expert 1)

> A good (piece of) mathematical writing should be able to convey the idea clearly…. . (Expert 3)

> The actual logical argument should be clear, … and (presented) in step by step manner. (Expert 3)

These views echo Steenrod (1973) who states that the organization of research should be separated into formal and informal structures: the former consists of definitions, theories and proofs, while the latter has introductory materials, which help to elucidate the mathematical discourse and includes motivations, analogies, examples, and metamathematical explanations. This division is conspicuously maintained in any mathematical presentation, so that readers can understand the work clearly.

A similar pattern is proposed by Kuteeva and McGrath (2015) who state that the epistemological process is invariably reflected in: the linear reasoning given by establishing the conditions, fixing notations, defining, contextualizing the background, making propositions, and stating theorems; and the meta-mathematical argument consisting of remarks and examples which are given to enhance the clarity of the reasoning.

In this discourse, higher importance is clearly accorded to content. Language is seen as second to content. Expert 1 argues that "there are good mathematics written with not so good skills" but he also concedes that "this is rare and phenomenal". Other experts are also of the view that language need not be perfect as long as it is understandable:

> Some mathematical papers have extremely long sentences and sometimes it is even hard to parse. (I made mistakes like that myself.) However, as long as the author can write clearly, it is ok for me. (Expert 2)

The secondary role of language in mathematical writing is also clear in Higham's (1998) description of how words should be employed:

> Use symbols if the idea would be too cumbersome to express in words or if it is important to make a precise mathematical statement… . Use words as long as they do not take up much more space than the corresponding symbols …. . Explain in words what the symbols mean if you think the reader might have difficulty grasping the meaning or essential feature. (p. 24)

Interaction with readers

The expert interviewees conceded that language is important for interacting with readers and for navigating them through mathematical discourse. Success in this will impact on whether the text engages readers' attention and interest. The experts commented thus:

> The author of a mathematical paper is an architect and good skills will help him to present a distinctive architecture. An artful and appealing facet of the building definitely attracts more visits for its interior content. (Expert 1)

> … English is the most commonly used language in the mathematical world. It cannot be avoided if you want any realistic interaction with other mathematicians. (Expert 3)

> Without good writing, the reader could lose patience easily, especially in this age where the internet is over flooded with information. For articles or books written by a really famous and established author, a general reader may have more patience, but otherwise, the writing may be easily passed over. (Expert 3)

Teaching

In the interviews, the experts indicated that teaching students how to write clearly and in good English belongs to the domain of the language teacher. While acknowledging the difficulty language teachers may have with the content, they sketched areas where the language teacher's contribution would be useful. In a follow-up interview, Expert 3 explained that he did not receive any formal instruction in mathematical writing. He picked up the skill by seeing how his supervisor wrote, and by being corrected by him:

> "It was a form of apprenticeship." (Expert 3)

He suggested modelling good mathematical writing as a useful teaching approach:

> It should be helpful … (for the teacher) to set some high standards of good writing by showing the works of some great mathematical writers. For instance, Jean-Pierre Serre and John Milnor set very good examples in their books and papers. (Expert 3)

In addition, getting students to plan before writing, giving feedback, facilitating peer reviews and going over multiple revisions with students have all been cited by the experts as effective teaching strategies. Despite these useful suggestions, however, the challenge for the language teacher, that content is sacrosanct, remains:

> The global logical structures of mathematical writing are highly specialized. It is near impossible for someone from another academic [area] to fully grasp what is said. Sometimes, even mathematicians working on other branches encounter difficulties in understanding. (Expert 4)

This same interviewee went on to suggest that language teachers should just focus on teaching English expressions for sections where natural language is likely to occur the most (e.g., the introduction chapter) and implied that the rest should be left to the subject specialist. However, the scope of language teaching should not be thus narrowed, given that natural language in mathematics is employed to also intricately highlight the epistemological process, engage the reader, and help readers recall inter- and intra-text sources. This highly specialised use of natural language is demonstrated by the findings of the corpus investigation discussed in the next section.

Results of the corpus investigation

An analysis of the corpus showed which metadiscourse categories were most frequently used, and how this pattern of metadiscourse usage differentiated from other disciplines. The items were ranked and compared with data taken from dissertations in applied linguistics, electronic engineering and computer science, which is ranked according to Hyland (2005, p. 57, see Table 2).

Table 2: Comparison of ranked items across disciplines (information for applied linguistics, electronic engineering and computer science drawn from Hyland, 2005).

	Mathematics (this study)	Applied Linguistics (Hyland, 2005)	Electronic Engineering (Hyland, 2005)	Computer Science (Hyland, 2005)
Interactive dimension				
Transitions	1	2	1	1
Frame markers	4	3	8	6
Endophoric markers	5	6	5	5
Evidentials	3	8	7	4
Code gloss	6	9	3	9
Interactional dimension				
Hedges	9	1	2	3
Boosters	8	10	10	10
Attitude markers	7	4	4	2
Self-mentions	10	5	9	8
Engagement markers	2	7	6	7

It is clear from Table 2 that apart from the almost identical ranking of transition across the four disciplines, mathematics metadiscourse use is, in many respects, clearly different from the other three disciplines. While engagement markers are ranked the second most employed resource in mathematical discourse, they rank much lower in the other disciplines. Another clear difference is in the use of hedges. While they rank near the bottom (rank: 9) in mathematics, they rank at or near the top in the other disciplines. To a lesser extent, mathematics discourse could also be differentiated from the other three disciplines in its use of attitude markers which ranks much lower (rank: 7) than applied science (rank: 4), electronic engineering (rank: 4) or computer science (rank: 2).

Table 3 shows the proportion of usage of Hyland's (2005) metadiscourse categories in the eight mathematics PhD theses. Clearly some of these categories are used more frequently than others. The ways they are used are discussed further below.

Table 3. Analysis of a corpus of eight mathematics PhD theses based on Hyland's (2005) metadiscourse model

Dimension	Category	Proportion of all occurrences %	Rank Order
Interactive	Transitions	32.60	1
	Frame markers	8.13	4
	endophoric markers	6.25	5
	Evidentials	11.41	3
	Code gloss	5.62	6
	(Subtotal)	64.01	
Interactional	Hedges	0.83	9
	Boosters	0.93	8
	Attitude markers	2.15	7
	Self-mentions	0.11	10
	Engagement markers	31.29	2
	(Subtotal)	35.31	
	(Total)	99.32	

Transition

Transition ranked as the most frequently used metadiscourse feature (32.6%) in this dataset which is consistent with theses in other disciplines (Hyland, 2005). In the samples studied, the transitions guide the reader through the logical reasoning using words, phrases and clauses that express relations mirroring the epistemological process. The types of transitions commonly found in the steps of reasoning are shown in Table 4 with examples.

Table 4. Types of transition found in the eight mathematical theses

Type of transition	Examples from the data
Addition	– **Moreover**, one can check easily that any morphism in X arises in this way, i.e. is equal to Y. – **In addition**, they showed that the SVT algorithm [17] is just one outer iteration of the exact primal PPA, … – **Additionally**, if the Slater condition (317) holds, the X is bounded.
Cause and effect	– **Therefore**, for each t > 0, one has …. – **It follows from** (3.3.) and (3.4) that … – **Thus**, we obtain the equivalence between (3.4) and (3.5). – Equation (3.6) immediately **implies** ….
Chronology	– **Later on**, Escobar considered the boundary value problem (1.3) in which … – **Now** we put .. (equation X).. **then** we have (equation Y) – **Next**, we prove the equivalence between (3.4) and (3.5).
Comparison and contrast	– **On the other hand**, Lemma 3.2 provides the lower bound: … – **Equivalently**, Equation X can also be obtained exactly via… – **In contrast**, we have (proof X) .. which is larger than the former upper bound (Proof y).
Giving examples	– Various results on the prescribing scalar curvature problem have been obtained during the past several decades. **One interesting study** is due to A. Chang and P. Yang [8] in 1991. – .. **for example**, the additive structure for the category X is given such that the sum of Y is by definition Z. – In many application, **such as** statistical regression and machine learning, f is a loss function which measures the difference between … – **For example** from 1 to 4, where the inner subproblem (3.34) is solved by the inexact smoothing Newton method…
Stating conditionals	– **Given** any T > 0, there exists a positive constant C = C (T), such that …. – **Suppose that** the solution set of problem (3.42) is nonempty and that Assumption 4 and 5 hold. – **Assume that** the sequence X generated by Algorithm sGS-PADMM is well defined. .. – Any section of X given by Y is a splitting **if and only if** the section Z produces a cocycle….

Imperatives

While transitions guide a reader through logical reasoning, explanations on how mathematical steps are performed are largely written in imperative statements which start directly with an infinitive verb (e.g. “recall”, “consider”). Underlying this way of “instructing” readers through the use of engagement markers (31.29%) is the assumption that they are actively involved in the steps. The writer is placing the responsibility of constructing the mathematical argument on the readers, while also taking the position that the reader is a member of the academic discourse community (Morgan, 1996). Below are some examples of how imperatives are used in the eight mathematical theses analysed in this study:

> **Recall** the definition of Baer sum.
>
> **Note** that in (ii) we do not require C which will enforce Y to be a homomorphism, which is unnecessary …
>
> **Choose** A. **Let** B be such that ….
>
> If E =0, then **stop.** Otherwise, **compute** C…
>
> **Solve** the following equation.
>
> **Replace** D and go to Step 1.

Use of reader reference (i.e. 'we' or 'us')

Academic writing communicates ideas but also persuades readers to accept propositional content. This is achieved through a dialogical process (Hyland, 2005), involving the assertion of the author's academic authority (and thus convincing the reader of the content's credibility), and engaging the reader in the logical reasoning process (Burton & Morgan, 2000; McGrath & Kuteeva, 2012; Morgan, 1996). In comparing the metadiscourse features in other disciplines (see Hyland, 2005) with that of pure mathematics, McGrath and Kuteeva (2012) note the significantly higher instances of engagement markers and attribute this to the extensive use of plural pronouns (i.e. "we" and "us") and imperatives. The data in the current study similarly showed that involvement with the reader is reflected primarily through the use of the pronoun "we" and imperatives, which coincides with the findings of previous studies (Burton & Morgan, 2000; McGrath & Kuteeva, 2012; Morgan, 1996). Below are some examples which illustrate the various uses of "we":

> **we** studied the problem of the existence of conformal metrics… (agentic we)
>
> **we** only partially achieve this goal, since we are not clear whether … (thinking process)

These examples clearly signal that it is the author(s), i.e. the expert, who is performing the process. However, there are instances where the "we" includes the reader (i.e. the "inclusive we"), with the expectation that the reader would re-perform some of the steps the author has done. For example:

> Observe that, if X, then with help of the divergence theorem, **we** also have Y. (inclusive "we")
>
> For the latter use, let **us** derive the flow equation for the mean curvature which is the following lemma.
>
> Now, **we** are ready to define a smoothing function for F(.).
>
> In order to prove the quadratic convergence of Algorithm 2, **we** need….

By using "we", the writer appears to be working through the steps with the reader and thereby engaging him in the process (Burton & Morgan, 2000). This coincides with the "invitational" tone that Su (2015) mentions. Such an approach is also intended to break the barrier between the author and his readers. Halmos (1973) likens this collegiality to an imaginary conversation with a friend on a long walk in the woods.

Evidentials and endophorics

Successful mathematical writers provide a tight interweaving of ideas involving many occurrences of cross referencing within the text and references to other works in the same field (Reiter, 1995). Endophoric markers (6.25% of the data set) are used to recall formulas and proofs in other parts of the writing to: set parameters, make assumptions, pre-empt a sequel to an earlier discussion in another part of the writing, state the basis for further action, and compare results with earlier findings. Here are some examples:

> **By Lemma 2.1**., we have that W is self-adjoint, which implies that V is self-adjoint.
>
> **From the above definition**, we know that …
>
> Then**, from (2.35),** we have …
>
> **By applying (2.34)** to formula X, we have equation Y.

It is also highly important to situate the research question within the mathematical discourse structure by referring to existing works. To do this, evidentials (11.41%) are employed when: considering the significance of the research work and situating it within the expert field; comparing the research results with existing results, theorems or hypotheses; discussing the equivalence of definitions; classifying theorems of structures or new proofs; connecting two previously unrelated aspects of mathematics; discussing a new method to an old problem; and putting forward a new proof for an old theorem (Reiter, 1995). For example:

> ***In their seminal paper, [BD01], Brylinski and Degline*** *studied a certain extension of G.*
>
> ***In [25], Chi, Funderlie and Plemmns*** *addressed some theoretical and numerical issues concerning structured low rank approximation problems.*
>
> *The problem (1.3) has many applications in diverse fields,* ***see [1, 2. 19. 37. 44. 82.102].***
>
> ***A tractable heuristic introduced in [36, 37]*** *is to minimize the nuclear norm over the same constraints as in (1.3) …*
>
> *Then* ***based on the famous result of Löwner [73],*** *we know that..*

Frame markers and code glosses

Frame markers (8.13%) are frequently used to help readers anticipate the contents of the next section or paragraph. They also define the structure of the thesis. Code glosses (5.62%) are used to explain or highlight the implications of definitions, theorems and results.

Attitude markers

Authorial identity, stance and attitude are shown through attitude markers (2.15%) which may be another way to engage the reader (Hyland, 2005). For example:

> It is **well known** that the prescribed mean curvature problem has a variational structure.
>
> For **simplicity**, we shall use the smoothing function X defined by (2.26).
>
> It is **relatively easy** to see that the key point is how to guarantee only one blow-up point without the simple bubble condition.

Such attitude markers reflect certain values and assumptions held by the academic community. The phrases "well-known" and "it is not uncommon to find" imply shared knowledge between the reader and the author. Acknowledging the readers' awareness of a certain fact indicates recognition of them as fellow academics. The allusion to simplicity and relative ease reflects a cultural value of mathematics that a straightforward, simple and easily understandable presentation is highly regarded.

Comments such as "the argument is easy to follow" and "beautiful construction" are well-understood by the discourse community to refer to the aesthetic value of the problem solving processes because of their structural clarity and brevity. According to Dreyfus and Eisenberg (1986), "clarity is easier to achieve with a simple argument than with a complicated one [although] brevity cannot always be measured in … words or pages; an important aspect of brevity is the number of logical steps and the step-size" (p. 3).

Hedges and boosters

Compared to attitude markers, hedges (0.83%) and boosters (0.93%) are much rarer. Hedges discredit or even downplay the validity of certain arguments and are used only to highlight the definitiveness of an important argument or a major contribution. Boosters are expressed through modals, chiefly with adjectives and adverbs. Here are some examples:

> The **only** possibility is that (adjective)
>
> … **only** finitely often (adjective)
>
> …. **never** has a change (adjective)
>
> It is **sufficient** to consider … (adjective)
>
> We have **successfully** built … (adverb)
>
> We **really** bring back… (adjective)
>
> … **definitely** stricter and larger than… (adverb)

Self-mentions

Self-mentions (i.e. the use of the pronoun "one" or "I") are rare. The mathematical process is a collaborative one and the mathematician is subordinate to the mathematics (Burton

& Morgan, 2000), therefore, self-mention is used only when the writer's persona is given importance.

The results show that, with the exception of engagement markers, interactive sub-categories are more often used than interactive sub-categories; and that much attention is given to structuring the discourse and ensuring that the reasoning is tightly interwoven.

Discussion

The data demonstrate that content is most import in mathematical writing. However, it is also evident that for a text to be accepted by the academic community, its content has to be conveyed in a clear, well-organized, aesthetically pleasing, culturally appropriate and engaging manner.

Clarity is achieved by organizing information into formal and informal parallel structures. The formal structure, consisting of the definitions, theorems, and proofs is essentially mathematical in nature, while the complementary informal or introductory material consisting of motivations, analogies, examples, and metamathematical explanations is where natural language occurs. Because PhD theses are written for a highly specialized audience (i.e. the examiner), whose deep knowledge of the subject is already assumed, code glosses may be omitted.

The mathematical work has to be situated in the larger schema of mathematical knowledge to gain credibility and authority and with reference to existing work to convince readers of the significance of contributions. To facilitate navigation through the internal structure of the text, metadiscourse features such as evidentials and endophoric markers are employed.

Transitions and engagement markers are the most frequently used devices in such texts because they guide the reader through the mathematical workings. Frame markers are used to help the reader anticipate the logical steps in the reasoning.

Finally, mathematical writing has a unique way of engaging readers and asserting authorial persona through use of pronoun references such as the "inclusive we" and the "authoritative we", and with directives and attitude markers.

Conclusion and pedagogical implications

The language teacher's input to the teaching of mathematical writing is hampered by a lack of specialized mathematical knowledge. Content and its organization are guided by the epistemological processes, and therefore is not within the domain of the language teacher. To a large extent too, mathematical logic can be understood only by mathematicians. Since EAP teachers are unlikely to be disciplinary specialists they must collaborate with disciplinary experts to understand propositional content, metalinguistic resources and disciplinary conventions. Jointly, they could develop materials relevant to mathematics PhD students. However, such collaborative opportunities can be challenging (see, for example, Simpson et. al., 2016 as cited in Li, Flowerdew and Cargill, 2018). Therefore, EAP teachers must acquire some specialized knowledge of the target discipline's culture, epistemological assumptions and genre (Cheng, 2015; Ferguson, 1997 as cited in Li, et. al., 2018).

An analysis, such as that provided by the present study, can help EAP teachers understand the metadiscourse features used to express disciplinary conventions. They may then use modelling, critical analysis and discussion of mathematical research texts to raise students' awareness of the necessary metadiscourse features which, according to Hyland (2005), helps students: understand the cognitive demands texts make on readers;

be better writers; and engage their readers in a way appropriate to their disciplinary community.

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About the author

Lee Ming Cherk is a senior lecturer at the Centre for English Language Communication of the National University of Singapore.

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Appendix 1: Sources consulted on "good mathematical" exposition

Title and Publication	**Genre**
How to Write Mathematics By N. E. Steenrod, P. R. Halmos, M. M. Schiffer and J. A. Dieudonne	Book published by the American Mathematical Society
Handbook of Writing for the Mathematical Sciences By N. J. Highams	Textbook published by the Society of Industrial and Applied Mathematics
A Primer of Mathematical Writing By S. G. Krantz	Textbook on mathematics
Writing a Research Paper in Mathematics By A. Reiter	Massachusetts Institute of Technology mathematics department website
Some Guidelines for Good Mathematical Writing By F. E. Su	MAA Focus, a newsletter published by the Mathematical Association of America
Mathematical Writing: A Brief Guide By M. Tomforde	University of Houston mathematics department website
Mathematical Writing By D. E. Knuth, T. Larrabee and P. M. Roberts	Transcription of a series of lectures on mathematical writing delivered at Stanford University

Appendix 2: Email to subject specialists requesting

Dear ,

How are you? My name is Lee Ming Cherk and I am from the Centre for English Language Communication at the National University of Singapore.

As a lecturer who has been teaching thesis writing to postgraduate students for more than 6 semesters, I have been fascinated by the unique way mathematics is written. As such, I have been doing research on this topic.

As such, I would be very grateful if you could help me by answering the following open-ended questions:

1. As a mathematician, how would you describe GOOD mathematical writing?

2. On a scale of 1-10, how important is good written English skills in mathematical writing? Please given reasons for your answer.

3. As a professor of mathematics, what do you think English instructors can do to help math majors and postgraduate students improve their mathematical writing skills?

The information you give will be most useful for future curriculum planning and course delivery. Also, once my paper is written up, I will be very happy to share the information with you. I look forward to hearing from you.

Thank you in advance and I wish you all the best in your endeavours.

Lee Ming Cherk
Senior Lecturer
Centre for English Language Communication